ISRAEL JOURNAL OF MATHEMATICS, Vol. 32, No. 4, 1979

ON PARTITIONS OF THE REAL LINE

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ABSTRACT
Answering a question of Sierpinski, we prove that the real line is not necessarily the disjoint union of $\aleph_1$ non-empty G_δ sets.

Introduction

In a beautiful paper ([1]), Hausdorff showed that $\{0,1\}^{\mathbf{N}}$, and therefore any uncountable Polish space, can be expressed as $\bigcup_{\xi<\aleph_1} E_\xi$, where each E_ξ is a G_δ set and the family $\langle E_\xi \rangle_{\xi<\aleph_1}$ is strictly increasing. It follows at once that the real line can be partitioned into $\aleph_1$ $F_{\sigma\delta}$ sets. The question naturally arises: can it be partitioned into $\aleph_1$ G_δ sets? (See [6], or [3], §39.II.) Of course it can if the continuum hypothesis is true; in this paper we shall show that there are models of set theory in which it cannot.

Our main result (Theorem 3) is, in effect, the following; If **R** can be partitioned into κ G_δ sets, where κ is uncountable, then **R** can be covered by κ nowhere dense closed sets. It follows immediately that **R** can be dissected into $\aleph_1$ G_δ sets iff it can be covered by $\aleph_1$ nowhere dense closed sets. But of course many models are known in which this is impossible.

In order to keep the main part of the argument free of appeals to special axioms, we write it in terms of the properties of a particular cardinal that has been studied elsewhere (e.g. [2]), the least cardinal of any family of nowhere dense closed sets covering **R**.

We should like to thank W. Fleissner, F. Galvin, L. Harrington, A. W. Miller and J. Stern for helpful correspondence and conversations. The second author would like to thank the National Science Foundation of the U.S.A. for partially supporting his research (Grant No. MCS 76–08479).

Received August 14, 1978 and in revised form February 1, 1979

0. Definitions

We begin with a word on terminology. For any set X, we write $\#(X)$ for the cardinal of X. We say that X is *countable* if $\#(X) \leqq \aleph_0$. We write $\mathfrak{c}$ for $\#(\mathbf{R})$. A *partition* of a set X is an unindexed disjoint family of subsets of X covering X.

A *Polish* space is a topological space in which the topology can be derived from a metric under which the space is separable and complete. A *Souslin* space is a Hausdorff topological space which is a continuous image of a Polish space. A topological space is *Baire* if the intersection of any sequence of dense open sets is dense.

If P is a partially ordered set, a subset Q of P is *downwards-cofinal* if for every $p \in P$ there is a $q \in Q$ such that $p \leqq q$.

1. The cardinal κ_0

Let κ_0 be the least cardinal of any family of nowhere dense closed sets covering $\mathbf{R}$. Then $\aleph_0 < \kappa_0 \leqq \mathfrak{c}$. We shall say that a G_Δ set (in any topological space) is one expressible as the intersection of fewer than κ_0 open sets.

We need some easy facts about κ_0.

2. Proposition

(a) *Let X be any non-empty Polish space without isolated points. Then κ_0 is the least cardinal of any family of nowhere dense closed sets covering X.*

(b) *If X is a Souslin space and $\mathscr{E}$ is a cover of X by closed sets such that $\#(\mathscr{E}) < \kappa_0$, then there is a countable $\mathscr{E}_0 \subseteq \mathscr{E}$ covering X.*

(c) *A G_Δ set in a Polish space is Baire in its induced topology.*

(d) *Let P be a non-empty, countable, partially ordered set, and $\mathscr{Q}$ a family of downwards-cofinal subsets of P such that $\#(\mathscr{Q}) < \kappa_0$. Then there is a totally ordered $P_0 \subseteq P$ meeting every $Q \in \mathscr{Q}$.*

Proof. (a) The point is that any such X has a dense G_δ set homeomorphic to $\mathbf{N}^{\mathbf{N}}$; so that any family of nowhere dense closed sets covering X gives rise to a family of nowhere dense closed sets covering $\mathbf{N}^{\mathbf{N}}$, and vice versa.

(b) Let Z be a Polish space and $f: Z \to X$ a continuous surjection. Let $\mathscr{D}$ be $\{f^{-1}[E] : E \in \mathscr{E}\}$. Let $\mathscr{G}$ be the collection of open subsets of Z that can be covered by a countable subfamily of $\mathscr{D}$. As Z has a countable base of open sets, $H = \bigcup \mathscr{G} \in \mathscr{G}$. Set $F = Z \setminus H$; then F is Polish. If $G \subseteq Z$ is open and $F \cap G \subseteq D$ for some $D \in \mathscr{D}$, then $G \subseteq H \cup D$ so $G \in \mathscr{G}$ and $F \cap G = \emptyset$; thus $F \cap D$ is nowhere dense in F for every $D \in \mathcal{D}$. As $F \subseteq \bigcup \mathcal{D}$, F can have no isolated points; as $\#(\mathcal{D}) < \kappa_0$, F must be empty, by part (a). Thus Z is covered by countably many sets from $\mathcal{D}$, and X is covered by countably many sets from $\mathcal{E}$.†

(c) Let Y be a G_Δ set in a Polish space, and $\langle G_n \rangle_{n \in \mathbb{N}}$ a sequence of dense subsets of Y which are open in the induced topology. Then $\bar{Y}$ is Polish, and $E = \bigcap_{n \in \mathbb{N}} G_n$ is expressible as the intersection of fewer than κ_0 dense open subsets of $\bar{Y}$. But now the complement of E in $\bar{Y}$ is the union of fewer than κ_0 nowhere dense closed sets, so cannot cover any non-empty open set in $\bar{Y}$, and E is dense in $\bar{Y}$, therefore dense in Y, as required.

(d) In the compact metric space $\mathcal{P}P$, let X be the set of maximal totally ordered subsets. Then X is G_δ being

$$\bigcap_{p \| q} \{t : t \subseteq P, p \notin t \text{ or } q \notin t\} \cap \bigcap_{p \in P} \bigcup_{q \| p} \{t : t \subseteq P, p \in t \text{ or } q \in t\},$$

where $p \| q$ if $p \not\leq q$ and $q \not\leq p$. So X, in its induced topology, is Polish.

For $Q \in \mathcal{Q}$, let G_Q be $\{t : t \in X_t, t \cap Q \neq \emptyset\}$. Then G_Q is an open set, and is also dense. For let G be a basic open set in X; then G is expressible as

$$\{t : t \in X, p_0, \cdots, p_n \in t, q_0, \cdots, q_m \notin t\}.$$

Fix $t \in G$; then there are $p'_0, \cdots, p'_m \in t$ such that $q_i \| p'_i$ for each $i \leqq m$ (because t is maximal). Let

$$p = \min(p_0, \cdots, p_n, p'_0, \cdots, p'_m),$$

and let $q \in Q$ be such that $q \leqq p$. Then there is a $u \in X$ such that

$$u \supseteq \{p_0, \cdots, p_n, p'_0, \cdots, p'_m, q\},$$

and $u \in G \cap G_Q$.

As $\#\{G_Q : Q \in \mathcal{Q}\} < \kappa_0$, there is a $t \in \bigcap_{Q \in \mathcal{Q}} G_Q$; this t is a totally ordered subset of P meeting every $Q \in \mathcal{Q}$.

3. Theorem

Let X be a Polish space, and $\mathcal{E}$ a partition of X into G_δ sets with $\#(\mathcal{E}) < \kappa_0$. Then $\mathcal{E}$ is countable.

REMARK. We can prove this theorem using forcing methods which is one of the ways it was done: As Miller [5] then used it in the consistency proof that the

†This part of the argument is used in [7].

existence of a partition of R to $\aleph_1$ G_δ sets does not imply the existence of a partition of R to $\aleph_1$ closed sets, the reader can look there.

PROOF. Suppose, if possible, otherwise.

(a) Let $\mathcal{I}$ be the σ-ideal of subsets of X covered by countably many sets from $\mathcal{E}$; the counter-hypothesis is that $X \notin \mathcal{I}$. Express each $E \in \mathcal{E}$ as $\bigcap_{n \in \mathbf{N}} G_E^n$ where each G_E^n is open. Enumerate a base for the topology of X as $\langle U_n \rangle_{n \in \mathbf{N}}$.

Let $\mathfrak{A}$ be the algebra of subsets of X generated by

$$\{U_n : n \in \mathbf{N}\} \cup \{G_E^n : n \in \mathbf{N}, E \in \mathcal{E}\}.$$

Then $\mathfrak{A}$ consists entirely of sets which are both F_σ and G_δ, and $\#(\mathfrak{A}) < \kappa_0$. Let $\mathfrak{A}_0 = \mathfrak{A} \cap \mathcal{I}$ and set $Y = X \setminus \bigcup \mathfrak{A}_0$; then Y is G_Δ.

(b) Observe that if $A \subseteq X$ is Souslin and $A \cap Y \in \mathcal{I}$, then $A \in \mathcal{I}$. For we have $A \cap Y \subseteq \bigcup \mathcal{E}_0$ for some countable $\mathcal{E}_0 \subseteq \mathcal{E}$; set $B = A \setminus \bigcup \mathcal{E}_0 \subseteq X \setminus Y = \bigcup \mathfrak{A}_0$. Now each element of $\mathfrak{A}_0$ is F_σ and $\#(\mathfrak{A}_0) < \kappa_0$, so the Souslin set B is covered by fewer than κ_0 closed sets all of which belong to $\mathcal{I}$. By Proposition 2(b) above, B is covered by countably many of these closed sets, so $B \in \mathcal{I}$ and $A \in \mathcal{I}$.

It follows (i) that $Y \neq \emptyset$, (ii) that if $B \in \mathfrak{A}$ and $B \cap Y \neq \emptyset$, then $B \cap Y \notin \mathcal{I}$.

(c) Define subalgebras $\mathfrak{B}_n$ of $\mathfrak{A}$ by

$$\mathfrak{B}_0 = \text{subalgebra generated by } \{U_n : n \in \mathbf{N}\},$$

$$\mathfrak{B}_{n+1} = \text{subalgebra generated by}$$

$$\mathfrak{B}_n \cup \{G_E^k : k \in \mathbf{N}, E \in \mathcal{E}, \exists B \in \mathfrak{B}_n, \overline{E \cap B \cap Y} \supseteq B \cap Y \neq \emptyset\}.$$

Then every $\mathfrak{B}_n$ is countable; this can be proved by induction, because if $B \in \mathfrak{A}$ and $B \cap Y \neq \emptyset$ then $B \cap Y$ is a G_Δ set, so $E \cap B \cap Y$ can be dense in $B \cap Y$ for at most one $E \in \mathcal{E}$, by Proposition 2(c). Consequently $\mathfrak{B} = \bigcup_{n \in \mathbf{N}} \mathfrak{B}_n$ is a countable subalgebra of $\mathfrak{A}$, containing every U_n, and with the property

$$\text{if } B \in \mathfrak{B}, E \in \mathcal{E} \text{ and } \overline{E \cap B \cap Y} \supseteq B \cap Y \neq \emptyset,$$

$$\text{then } G_E^k \in \mathfrak{B} \text{ for every } k \in \mathbf{N}.$$

(d) We now use Proposition 2(d). Let P be

$$\{\overline{B \cap Y} : B \in \mathfrak{B}, B \cap Y \neq \emptyset\},$$

ordered by inclusion. Let

$$Q^n = \{F : F \in P, \operatorname{diam}(F) \leqq 2^{-n}\},$$

$$Q_E = \{F : F \in P, F \cap E = \emptyset\} \quad \text{for } E \in \mathscr{E}.$$

To see that these are downwards-cofinal, argue as follows:

(i) If $F \in P$ and $n \in \mathbf{N}$, express F as $\overline{B \cap Y}$ where $B \in \mathfrak{B}$. Let k be such that $\operatorname{diam}(U_k) \leqq 2^{-n}$ and $B \cap Y \cap U_k \neq \emptyset$; now $B \cap U_k \in \mathfrak{B}$, so $F_1 = \overline{B \cap Y \cap U_k} \in P$ and $F_1 \subseteq F$, $F_1 \in Q^n$.

(ii) If $F \in P$ and $E \in \mathscr{E}$, express F as $B \cap Y$ where $B \in \mathfrak{B}$. Consider two cases separately:

(α) $E \cap B \cap Y$ is dense in $B \cap Y$. In this case, $G_E^k \in \mathfrak{B}$ for every $k \in \mathbf{N}$. But also $E \not\supseteq B \cap Y$ because $B \cap Y \notin \mathscr{I}$, by part (b), so there is some $k \in \mathbf{N}$ such that $B \cap Y \backslash G_E^k \neq \emptyset$. Now $B \backslash G_E^k \in \mathfrak{B}$, so $F_1 = \overline{(B \backslash G_E^k) \cap Y} \in P$, $F_1 \subseteq F$, $F_1 \cap E \subseteq F_1 \cap G_E^k = \emptyset$, so $F_1 \in Q_E$.

(β) $E \cap B \cap Y$ is not dense in $B \cap Y$. In this case (because $B \cap Y$ is a G_δ set, therefore Baire, by Proposition 2(c)), there is a $k \in \mathbf{N}$ such that $G_E^k \cap B \cap Y$ is not dense in $B \cap Y$. Let $r \in \mathbf{N}$ be such that $G_E^k \cap B \cap Y \cap U_r = \emptyset$, but $B \cap Y \cap U_r \neq \emptyset$. Now we find that $F_1 = \overline{B \cap U_r \cap Y} \in P$, $F_1 \subseteq F$, and $F_1 \in Q_E$.

(e) Accordingly, by Proposition 2(d) above, there is a totally ordered $P_0 \subseteq P$ meeting every Q^n and every Q_E. Because X is Polish and P_0 meets every Q^n, $\bigcap P_0$ is a singleton $\{t\}$; now $t \notin \bigcup \mathscr{E}$, because P_0 meets every Q_E.

This is the required contradiction.

4. Corollary

Let X be a Souslin space and $\mathscr{E}$ a partition of X into $G_{\delta\sigma}$ sets with $\#(\mathscr{E}) < \kappa_0$. Then $\mathscr{E}$ is countable.

Proof. Since X is a continuous image of a Polish space, it is enough to consider the case in which X is itself Polish. In this case every $G_{\delta\sigma}$ set can be dissected into countably many G_δ sets ([3], §30.V, theorem 2), so X can be dissected into a family of G_δ sets of cardinality not greater than $\max(\aleph_0, \#(\mathscr{E}))$, but at least $\#(\mathscr{E})$, and $\mathscr{E}$ must be countable.

5. Conclusions

It follows that if $\kappa_0 > \aleph_1$, then no Polish space can be partitioned into $\aleph_1$ non-empty $G_{\delta\sigma}$ sets. Obviously the converse is true: if $\kappa_0 = \aleph_1$, then $\mathbf{R}$ can be dissected into $\aleph_1$ G_δ sets. (But it does not seem to follow that $\mathbf{R}$ can always be dissected into κ_0 G_δ sets, though of course it can be dissected into κ_o G_Δ sets.) Another way of phrasing Theorem 3 is to say: if **R** can be dissected into κ G_δ sets, where κ is uncountable, then $\kappa \geqq \kappa_0$, so that **R** can be covered by κ nowhere dense closed sets.

Martin's Axiom implies that $\kappa_0 = \mathfrak{c}$ ([4], §4); so MA + not-CH implies that $\kappa_0 > \aleph_1$. Similarly, $\kappa_0 = \mathfrak{c}$ in any model obtained from a model of CH by adding mutually generic Cohen reals. On the other hand, $\kappa_0 = \aleph_1$ in any model obtained from a model of CH by adding random reals. In these models, Kunen has shown that **R** can be dissected into $\aleph_1$ closed sets (see [7]); but A. W. Miller has found a model in which $\kappa_0 = \aleph_1$ and $\mathbf{R}_1$ cannot be dissected into $\aleph_1$ closed sets ([5]).

In [2], models are constructed in which $\kappa_0 = \mathrm{cf}(\mathfrak{c})$ but is otherwise unrestricted. A. W. Miller has shown (private communication) that $\mathrm{cf}(\kappa_0) > \aleph_0$.

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